1 Combinatorial Proofs

The binomial coefficient is introduced as the number of ways of choosing k distinct objects as a subset from n distinct objects. Many relationships exist between binomial coefficients and these results were proven using algebraic methods. These algebraic manipulations did not motivate us as to why anyone would have thought of these relationships in the first place. The algebraic manipulations just pushed symbols around without appealing to their underlying structure as a counting tool.

In contrast, the combinatorial proof is based on the concept that there is fundamentally more to the truth of an equation than just the algebraic manipulation of symbols. The relationship expressed by the equation exists because some set S has been counted in two different ways. Since the number of objects in the set S remains constant, the two different methods used must produce equal results. The beauty, richness and elegance of the combinatorial proof lies in understanding why two quantities are equal. Such understanding is lacking in algebraic manipulations.

Theorem 1 $\binom{n}{k} = \binom{n}{n-k}$

Proof. *Let S be the collection of all k element subsets of an n element set. One the one hand the definition of the binomial coefficient gives the number of elements in S as $\binom{n}{k}$. On the other hand, from the n elements select $n-k$ elements to not appear. The remaining k elements form the desired k element subset. Note that each different set of $n-k$ elements produces a different k element subset. Both approaches count the same set S and must be equal.*

■

Of course it can be difficult determining the set that is counted in two different ways. One strategy is to look at the simpler side of the equation to help formulate the set. Consider $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Using the left hand side of the equation we need to let S be the collection of all k element subsets of an n element set. Since this is the same set we used to show $\binom{n}{k} = \binom{n}{n-k}$ it must be the alternate way of counting that provides the right hand side of the equation. Analysis indicates that the selections are made from a smaller set than the original n elements. In this theorem, it appears that a single element is excluded.

Theorem 2 $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Proof. Let S be the collection of all k element subsets of an n element set. One the one hand the definition of the binomial coefficient gives this as $|S| = \binom{n}{k}$. On the other hand, partition the k element subsets into two disjoint cases based on a specific element x. Case 1 counts those k element subsets that contain the element x. Since x is included in the subset, select the other $k-1$ elements for the subset from the remaining $n-1$ elements. This can be done in $\binom{n-1}{k-1}$ ways. Case 2 counts those k element subsets that do not contain the element x. Because x is not included in the k element subset we must select all elements from the remaining $n-1$ objects. This can be done in $\binom{n-1}{k}$ ways. The two cases are disjoint and contain all possible k element subsets so the sum rule applies and $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$. Both approaches count the size of the same set S and equality follows. ■

As the identities become more complex, more notation and explicit construction of sets may be needed for clarity.

Theorem 3 *For all non-negative integers n,* $\binom{2n}{2} = n^2 + 2\binom{n}{2}$.

Proof. Let $A = \{1, 2, ..., n-1, n, n+1, ..., 2n\}$ and partition A into the two subsets $B = \{1, 2, ..., n\}$ and $C = \{n+1, n+2, ..., 2n\}$. Let S be the collection of all two element subsets of A. Immediately we see that $|S| = \binom{2n}{2}$. On the other hand we can think of selection two elements from A in terms of its subsets B and C. Three possibilities exist. Both elements could be selected from B in $\binom{n}{2}$ ways. Similarly, both elements could be selected from C in $\binom{n}{2}$ ways. Finally we might select one element from each of B and C. This selection could be made in $n * n = n^2$ ways. These three ways cover all selections of two elements from A based on its partitioning into B and C. Thus, $|S| = 2\binom{n}{2} + n^2$. ■

All theorems with identities involving a sum should be broken down into disjoint cases and an application of the sum rule be applied. What about an identity that involves a product? The multiplication rule will need to be invoked and the set to be counted in two different ways must be ordered.

Theorem 4 *For all non-negative integers n and k,* $\binom{n}{k}\binom{k}{2} = \binom{n}{2}\binom{n-2}{k-2}$.

Proof. Here we are looking at nested subsets. $A = \{1, 2, ..., n\}$. Let S be a collection of ordered pairs (B, C) such that B is a subset of A, C is a subset of B, $|B| = k$ and $|C| = 2$. On the one hand, the size of S can be determined by first selecting B and then selecting C. With no restrictions, B can be selected in $\binom{n}{k}$ ways. Now C must be selected from the k elements of B which can be done in $\binom{k}{2}$ ways. Hence, there are $\binom{n}{k}\binom{k}{2}$ different ordered pairs in S. On the one hand, the size of S can also be determined by first selecting C and then building B up from C. With no restrictions, C can be selected in $\binom{n}{2}$ ways. Now, B must be built up from C. The set B needs $k-2$ additional elements from the remaining $n-2$ original elements. So, B can now be selected in $\binom{n-2}{k-2}$ ways. Thus, the size of S is also $\binom{n}{2}\binom{n-2}{k-2}$. Counting $|S|$ in two ways shows $\binom{n}{k}\binom{k}{2} = \binom{n}{2}\binom{n-2}{k-2}$. ■

Let's return to the Binomial Theorem. The most intuitive proof of the Binomial Theorem is combinatorial.

Theorem 5 *For any real values x and y and non-negative integer n,* $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof. Each term in the expansion of $(x+y)^n$ will be of the form $k_i x^i y^{n-i}$ where k_i is some coefficient. How often the expansion of $(x+y)^n$ yield an $x^i y^{n-i}$

term? Exactly i of the x terms must be selected from the n products of $x+y$. Of course, this automatically determines which of the y terms will be selected. The selection of the x terms can be done in $\binom{n}{i}$ ways. Hence, $k_i = \binom{n}{i}$.[1] ■

Theorem 6 $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

Prove each of the following for all non-negative integer values (unless otherwise stated) using a combinatorial proof.

Exercise 7 $\binom{n}{1} = n$

Exercise 8 $\binom{n}{n-1} = n$

Exercise 9 $\binom{n}{n} = 1$

Exercise 10 $\binom{n+m}{2} = nm + \binom{n}{2} + \binom{m}{2}$

Exercise 11 $\binom{n}{3} = \binom{n-2}{3} + 2\binom{n-2}{2} + n - 2$ *for* $n \geq 5$.

Exercise 12 $k\binom{n}{k} = n\binom{n-1}{k-1}$

Exercise 13 $(n-k)\binom{n}{k} = n\binom{n-1}{k}$

Exercise 14 $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$

Exercise 15 $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ *That is, generalize Theorem 4.*

Exercise 16 $n^3 = 3!\binom{n}{3} + 3!\binom{n}{2} + n$

Exercise 17 $\binom{3n}{2} = 3\binom{n}{2} + 3n^2$

Exercise 18 $\binom{3n}{3} = n^3 + 6n\binom{n}{2} + 3\binom{n}{3}$

Exercise 19 $\binom{n}{m}\binom{n-m}{k} = \binom{n}{m+k}\binom{m+k}{m}$

Exercise 20 $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Exercise 21 $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

[1] This is no surprise. After all, if $\binom{n}{k}$ is called the binomial coefficient then it had better be the coefficient of something.